The emergence of the Virasoro and w_∞ algebras through the renormalized powers of quantum white noise

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Abstract

We introduce a new renormalization for the powers of the Dirac delta function. We show that this new renormalization leads to a second quantized version of the Virasoro sector w_∞ of the extended conformal algebra with infinite symmetries W_∞ of Conformal Field Theory ([4]-[7], [11], [13], [14]). In particular we construct a white noise (boson) representation of the w_∞ generators and commutation relations and of their second quantization.

1 Introduction

Classical (i.e Itô [10]) and quantum (i.e Hudson-Parthasarathy [12]) stochastic calculi were unified by Accardi, Lu and Volovich in [3] in the framework of Hida's white noise theory by expressing the fundamental noise processes in terms of the Hida white noise functionals b_t and $b_t^\dagger$ defined as follows: Let $L^2_{sym}(\mathbb{R}^n)$ denote the space of square integrable functions on $\mathbb{R}^n$ which are symmetric under permutation of their arguments and let $\mathcal{F} := \bigoplus_{n=0}^\infty L^2_{sym}(\mathbb{R}^n)$ where if $\psi := \{\psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}$, then $\psi^{(0)} \in \mathbf{C}$, $\psi^{(n)} \in L^2_{sym}(\mathbb{R}^n)$ and

$$\|\psi\|^2 = \|\psi(0)\|^2 + \sum_{n=1}^\infty \int_{\mathbb{R}^n} |\psi^{(n)}(s_1,\dots,s_n)|^2 ds_1 \dots ds_n$$

The subspace of vectors $\psi = \{\psi^{(n)}\}_{n=0}^\infty \in \mathcal{F}$ with $\psi^{(n)} = 0$ for all but finitely many n will be denoted by $\mathcal{D}_0$. Denote by $S \subset L^2(\mathbb{R}^n)$ the Schwartz space of smooth functions decreasing at infinity faster than any polynomial and let $\mathcal{D}$ be the set of all $\psi \in \mathcal{F}$ such that $\psi^{(n)} \in S$ and $\sum_{n=1}^\infty n\,|\psi^{(n)}|^2 < \infty$. For each $t \in \mathbb{R}$ define the linear operator $b_t : \mathcal{D} \to \mathcal{F}$ by

$$(b_t\psi)^{(n)}(s_1,\dots,s_n) := \sqrt{n+1}\psi^{(n+1)}(t,s_1,\dots,s_n)$$

and the operator valued distribution $b_t^\dagger$ by

$$(b_t^\dagger\psi)^{(n)}(s_1,\dots,s_n) := \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\delta(t-s_i)\psi^{(n-1)}(s_1,\dots,\hat{s}_i,\dots,s_n)$$

where δ is the Dirac delta function and $\hat{}$ denotes omission of the corresponding variable. The white noise functionals satisfy the Boson commutation relations

$$[b_t, b_s^\dagger] = \delta(t-s)$$

$$[b_t^\dagger, b_s^\dagger] = [b_t, b_s] = 0$$

and the duality relation

$$(b_s)^* = b_s^\dagger$$

Letting $\mathcal{H}$ be a test function space we define for $f \in \mathcal{H}$ and $n,k \in \{0,1,2,...\}$ the sesquilinear form on $\mathcal{D}_0$

$$B_k^n(f) := \int_{\mathbb{R}} f(t)\, {b_t^\dagger}^n\, b_t^k\, dt$$

i.e for ϕ, ψ in $\mathcal{D}_0$ and $n,k \geq 0$

$$< \psi, B_k^n(f)\,\phi >= \int_{\mathbb{R}} f(t)\ < b_t^n\,\psi, b_t^k\,\phi >\ dt$$

with involution

$$(B_k^n(f))^* = B_n^k(\bar{f})$$

and with

$$B_0^0(\bar{g}f) = \int_{\mathbb{R}} \bar{g}(t)\, f(t)\, dt =< g, f >$$

The Fock representation is characterized by the existence of a unit vector Φ, called the Fock vacuum vector, cyclic for the operators $B_n^k(\bar{f})$ and satisfying:

$$B_k^0\Phi = B_k^h\Phi = 0 \qquad ; \qquad \forall k > 0\ ;\ \forall h \geq 0 \tag{1.1}$$

It is not difficult to prove that, if the Fock representation exists, it is uniquely characterized by the two above mentioned properties.

In [1] it was proved that for all $t,s \in \mathbb{R}_+$ and $n,k,N,K \geq 0$, one has:

$$[{b_t^\dagger}^n b_t^k, {b_s^\dagger}^N b_s^K] = \tag{1.2}$$

$$\epsilon_{k,0}\epsilon_{N,0}\sum_{L\geq 1}\binom{k}{L}N^{(L)}\,b_t^{\dagger\,n}\,b_s^{\dagger\,N-L}\,b_t^{k-L}\,b_s^K\,\delta^L(t-s)$$

$$-\epsilon_{K,0}\epsilon_{n,0}\sum_{L\geq 1}\binom{K}{L}n^{(L)}\,b_s^{\dagger\,N}\,b_t^{\dagger\,n-L}\,b_s^{K-L}\,b_t^k\,\delta^L(t-s)$$

where

$$\epsilon_{n,k} := 1-\delta_{n,k}$$

$\delta_{n,k}$ is Kronecker's delta and the decreasing factorial powers $x^{(y)}$ are defined by

$$x^{(y)} := x(x-1)\cdots(x-y+1)$$

with $x^{(0)}=1$. In order to consider higher powers of b_t and $b_t^\dagger$, the renormalization

$$\delta^l(t) = c^{\,l-1}\,\delta(t), \quad l=2,3,.... \tag{1.3}$$

where $c>0$ is an arbitrary constant, was introduced in [3]. Then (1.2) becomes

$$[b_t^{\dagger\,n}b_t^k, b_s^{\dagger\,N}b_s^K] = \tag{1.4}$$

$$\epsilon_{k,0}\epsilon_{N,0}\sum_{L\geq 1}\binom{k}{L}N^{(L)}\,c^{\,L-1}\,b_t^{\dagger\,n}\,b_s^{\dagger\,N-L}\,b_t^{k-L}\,b_s^K\,\delta(t-s)$$

$$-\epsilon_{K,0}\epsilon_{n,0}\sum_{L\geq 1}\binom{K}{L}n^{(L)}\,c^{\,L-1}\,b_s^{\dagger\,N}\,b_t^{\dagger\,n-L}\,b_s^{K-L}\,b_t^k\,\delta(t-s)$$

Multiplying both sides of (1.4) by test functions $f(t)\bar{g}(s)$ and formally integrating the resulting identity (i.e. taking $\int\int\ \dots\ dsdt$), we obtain the following commutation relations for the Renormalized Powers of Quantum White Noise (RPQWN)

$$[B_K^N(\bar{g}), B_k^n(f)] \tag{1.5}$$

$$=\sum_{L=1}^{K\wedge n} b_L(K,n)\,B_{K+k-L}^{N+n-L}(\bar{g}f) - \sum_{L=1}^{k\wedge N} b_L(k,N)\,B_{K+k-L}^{N+n-L}(\bar{g}f)$$

$$=\sum_{L=1}^{(K\wedge n)\vee(k\wedge N)} \theta_L(N,K;n,k)\,c^{L-1}\,B_{K+k-L}^{N+n-L}(\bar{g}f)$$

where

$$b_x(y,z) := \epsilon_{y,0}\,\epsilon_{z,0}\binom{y}{x}\,z^{(x)}\,c^{x-1} \tag{1.6}$$

and for $n, k, N, K \in \{0, 1, 2, ...\}$

$$\theta_L(N, K; n, k) := \epsilon_{K,0}\, \epsilon_{n,0} \binom{K}{L} n^{(L)} - \epsilon_{k,0}\, \epsilon_{N,0} \binom{k}{L} N^{(L)} \tag{1.7}$$

with

$$\sum_{L=1}^{(K\wedge n)\vee(k\wedge N)} = 0$$

if $(K \wedge n) \vee (k \wedge N) = 0$. In what follows we will use the notation

$$B^n_k := B^n_k(\chi_I) \tag{1.8}$$

whenever $I \subset \mathbb{R}$ with $\mu(I) < +\infty$ is fixed. Moreover, to simplify the notations, we will use the same symbol for the generators of the RPQWN algebra and for their images in a given representation. As above, we denote by Φ the Fock vacuum vector with $b_t\, \Phi = 0$ and $\langle \Phi, \Phi \rangle = 1$. It was proved in [1] that, with commutation relations (1.5), the B^n_k do not admit a common Fock space representation. The main counter-example is that if a common Fock representation of the B^n_k existed, one should be able to define inner products of the form

$$< (a\, B^{2n}_0(\chi_I) + b\, (B^n_0(\chi_I))^2)\Phi, (a\, B^{2n}_0(\chi_I) + b\, (B^n_0(\chi_I))^2)\Phi >$$

where $a, b \in \mathbb{R}$ and I is an arbitrary interval of finite measure $\mu(I)$. Using the notation $< x >=< \Phi, x\, \Phi >$ this amounts to the positive semi-definiteness of the quadratic form

$$a^2\ < B^0_{2n}(\chi_I)\, B^{2n}_0(\chi_I) > +2\, a\, b\ < B^0_{2n}(\chi_I)(B^n_0(\chi_I))^2 >$$

$$+\ b^2\ < (B^0_n(\chi_I))^2\, (B^n_0(\chi_I))^2 >$$

or equivalently of the (2×2) matrix

$$A = \begin{bmatrix} < B^0_{2n}(\chi_I)\, B^{2n}_0(\chi_I) > & < B^0_{2n}(\chi_I)\, (B^n_0(\chi_I))^2 > \\ < B^0_{2n}(\chi_I)\, (B^n_0(\chi_I))^2 > & < (B^0_n(\chi_I))^2\, (B^n_0(\chi_I))^2 > \end{bmatrix}$$

Using the commutation relations (1.5) we find that

$$A = \begin{bmatrix} (2n)!c^{2n-1}\mu(I) & (2n)!c^{2n-2}\mu(I) \\ (2n)!c^{2n-2}\mu(I) & 2(n!)^2c^{2n-2}\mu(I)^2 + \left((2n)! - 2(n!)^2\right) c^{2n-3}\mu(I) \end{bmatrix}$$

The matrix A is symmetric, so it is positive semi-definite only if its minors are non-negative. The minor determinants of A are

$$d_1 = (2n)!c^{2n-1}\mu(I) \geq 0$$

and

$$d_2 = 2c^{4(n-1)}\mu(I)^2(n!)^2(2n)!(c\,\mu(I)-1) \geq 0 \Leftrightarrow \mu(I) \geq \frac{1}{c}.$$

Thus the interval I cannot be arbitrarily small. The counter-example was extended in [2] to the q-deformed case

$$b_t\, b_s^\dagger - q\, b_s^\dagger\, b_t = \delta(t-s)$$

A stronger no-go theorem, which establishes the impossibility of a Fock representation of any Lie algebra containing B_0^n for any $n \geq 3$ and satisfying commutation relations (1.5), can be proved using the following results.

Lemma 1. *Let $n \geq 3$ and define*

$$C_1(n) := [B_n^0, B_0^n]$$

and for $k \geq 2$

$$C_k(n) := [B_n^0, C_{k-1}(n)].$$

Then

$$C_3(n) = \beta(n)\, B_{2n}^0 + N(n) \tag{1.9}$$

where, in the notation (1.6), $\beta(n) \in \mathbb{R} - \{0\}$ is given by

$$\beta(n) := \sum_{L_1=1}^{n-1} \sum_{L_2=1}^{n-L_1} b_{L_1}(n,n)\, b_{L_2}(n,n-L_1)\, b_{n-(L_1+L_2)}(n,n-(L_1+L_2)) \tag{1.10}$$

and $N(n)$ is a sum of operators given by

$$N(n) := \sum_{L_1=1}^{n-1} \sum_{L_2=1}^{n-L_1} \sum_{L_3=1}^{n-(L_1+L_2)} b_{L_1}(n,n)\, b_{L_2}(n,n-L_1) \tag{1.11}$$

$$\times b_{L_3}(n,n-(L_1+L_2))B_{3n-(L_1+L_2+L_3)}^{n-(L_1+L_2+L_3)}$$

with adjoint

$$N(n)^* := \sum_{L_1=1}^{n-1} \sum_{L_2=1}^{n-L_1} \sum_{L_3=1}^{n-(L_1+L_2)} b_{L_1}(n,n)\, b_{L_2}(n,n-L_1) \tag{1.12}$$

$$\times\, b_{L_3}(n,n-(L_1+L_2))B_{n-(L_1+L_2+L_3)}^{3n-(L_1+L_2+L_3)}$$

where the triple summations in (1.11) and (1.12) are over all L_1, L_2, L_3 such that $L_1 + L_2 + L_3 \neq n$.

Proof. The commutation relations (1.5) imply that:

$$C_1(n) = [B_n^0, B_0^n] = \sum_{L_1=1}^{n} b_{L_1}(n,n)\, B_{n-L_1}^{n-L_1}$$

and

$$C_2(n) = [B_n^0, C_1(n)] = \sum_{L_1=1}^{n} b_{L_1}(n,n)\, [B_n^0, B_{n-L_1}^{n-L_1}]$$

$$= \sum_{L_1=1}^{n} \sum_{L_2=1}^{n-L_1} b_{L_1}(n,n)\, b_{L_2}(n, n-L_1)\, B_{2n-(L_1+L_2)}^{n-(L_1+L_2)}$$

$$= \sum_{L_1=1}^{n-1} \sum_{L_2=1}^{n-L_1} b_{L_1}(n,n)\, b_{L_2}(n, n-L_1)\, B_{2n-(L_1+L_2)}^{n-(L_1+L_2)}$$

since $[B_n^0, B_{n-L_1}^{n-L_1}] = 0$ for $L_1 = n$, and finally

$$C_3(n) = [B_n^0, C_2(n)]$$

$$= \sum_{L_1=1}^{n-1} \sum_{L_2=1}^{n-L_1} b_{L_1}(n,n)\, b_{L_2}(n, n-L_1)\, [B_n^0, B_{2n-(L_1+L_2)}^{n-(L_1+L_2)}]$$

$$= \sum_{L_1=1}^{n-1} \sum_{L_2=1}^{n-L_1} \sum_{L_3=1}^{n-(L_1+L_2)} b_{L_1}(n,n)\, b_{L_2}(n, n-L_1)$$

$$\times b_{L_3}(n, n-(L_1+L_2))\, B_{3n-(L_1+L_2+L_3)}^{n-(L_1+L_2+L_3)}$$

from which (3.2) follows by splitting the above triple sum into the parts $L_1 + L_2 + L_3 = n$ and $L_1 + L_2 + L_3 \neq n$. □

Remark 1.

Notice that $3n - (L_1 + L_2 + L_3)$ is at least equal to $2n$ and

$$N(n)^*\, \Phi := \sum_{L_1=1}^{n-1} \sum_{L_2=1}^{n-L_1} \sum_{L_3=1}^{n-(L_1+L_2)} b_{L_1}(n,n)\, b_{L_2}(n, n-L_1)$$

$$\times b_{L_3}(n, n-(L_1+L_2)) B_{n-(L_1+L_2+L_3)}^{3n-(L_1+L_2+L_3)} \Phi = 0$$

due to (1.1) and $n - (L_1 + L_2 + L_3) \neq 0$.

Remark 2.

For $n = 2$ the previous lemma is not valid since

$$C_1(2) = 2\,B_0^0 + 4\,B_1^1,\;\; C_2(2) = 8\,B_2^0,\;\; C_3(2) = 0 \Rightarrow \beta(2) = 0$$

Therefore, what follows is not in contradiction with the well established Fock representation of the square of white noise operators B_0^2, B_2^0 and B_1^1 proved in [3].

Corollary 1. *Let $n \geq 3$ and suppose that an operator $*$-Lie sub algebra $\mathcal{L}$ of the RPQWN algebra contains B_0^n. Then $\mathcal{L}$ will also contain*

$$a\,\left(\beta(n)\,B_0^{2n} + N(n)^*\right) + b\,(B_0^n)^2$$

for all $a, b \in \mathbb{R}$, where $\beta(n)$ and $N(n)^*$ are as in (1.10) and (1.12) respectively.

Proof. Since $\mathcal{L}$ is an operator algebra containing B_0^n, it will also contain $(B_0^n)^2$ and $b\,(B_0^n)^2$. By the $*$-property $\mathcal{L}$ will also contain B_n^0 and since $\mathcal{L}$ is a Lie algebra, by lemma 1 , it will contain $\beta(n)\,B_{2n}^0 + N(n)$ and $a\,\left(\beta(n)\,B_{2n}^0 + N(n)\right)$. Again by the $*$-property, $\mathcal{L}$ will contain $a\,\left(\beta(n)\,B_0^{2n} + N(n)^*\right)$ and, since $\mathcal{L}$ is a vector space, it will also contain $a\,\left(\beta(n)\,B_0^{2n} + N(n)^*\right) + b\,(B_0^n)^2$. $\square$

Theorem 1. *Let $n \geq 3$ and suppose that an operator $*$-Lie sub algebra $\mathcal{L}$ of the RPQWN algebra contains B_0^n. Then $\mathcal{L}$ does not admit a Fock space representation.*

Proof. By Corollary 1, $\mathcal{L}$ will also contain $a\,\left(\beta(n)\,B_0^{2n} + N(n)^*\right) + b\,(B_0^n)^2$, for all $a, b \in \mathbb{R}$, where $\beta(n)$, $N(n)^*$ are as in (1.10) and (1.12) respectively. As in the previously discussed counter-example, it follows that the Fock-vacuum norm

$$\|\left(a\,\left(\beta(n)\,B_0^{2n} + N(n)^*\right) + b\,(B_0^n)^2\right)\Phi\| = \|\left(a\,B_0^{2n} + b\,(B_0^n)^2\right)\Phi\|$$

cannot be nonnegative for arbitrarily small $I \subset \mathbb{R}$. $\square$

In the remaining sections of this paper we provide a new renormalization prescription for the powers of the delta function which bypasses the no-go theorems proved so far and which leads to an unexpected connection with the Virasoro algebra and the w_∞ and W_∞ algebras of Conformal Field Theory (cf. [11]).

2 A new look at the counter-example of the previous section

In this section we generalize (1.3) to

$$\delta^l(t-s) = \phi^{l-1}(s)\,\delta(t-s), \quad l = 2, ... \tag{2.1}$$

and we look for conditions on $\phi(s)$, and an appropriate set of test functions, that eliminate the difficulties posed by the counter-example of Section 1. The white noise commutation relations (1.2) now become

$$[{b_t^\dagger}^n b_t^k, {b_s^\dagger}^N b_s^K] = \tag{2.2}$$

$$\epsilon_{k,0}\epsilon_{N,0}\sum_{L\geq 1}\binom{k}{L} N^{(L)}\, {b_t^\dagger}^n\, {b_s^\dagger}^{N-L}\, b_t^{k-L}\, b_s^K\, \phi^{L-1}(s)\,\delta(t-s)$$

$$-\epsilon_{K,0}\epsilon_{n,0}\sum_{L\geq 1}\binom{K}{L} n^{(L)}\, {b_s^\dagger}^N\, {b_t^\dagger}^{n-L}\, b_s^{K-L}\, b_t^k\, \phi^{L-1}(s)\,\delta(t-s)$$

from which, by multiplying both sides by $f(t)\bar{g}(s)$ and integrating the resulting identity we obtain

$$[B_K^N(\bar{g}), B_k^n(f)] = \tag{2.3}$$

$$\sum_{L=1}^{K\wedge n} \hat{b}_L(K,n)\, B_{K+k-L}^{N+n-L}(\bar{g}\, f\, \phi^{L-1}) - \sum_{L=1}^{k\wedge N} \hat{b}_L(k,N)\, B_{K+k-L}^{N+n-L}(\bar{g}\, f\, \phi^{L-1})$$

$$= \sum_{L=1}^{(K\wedge n)\vee(k\wedge N)} \theta_L(N,K;n,k)\, B_{K+k-L}^{N+n-L}(\bar{g}\, f\, \phi^{L-1})$$

where

$$\hat{b}_x(y,z) := \epsilon_{y,0}\,\epsilon_{z,0}\binom{y}{x} z^{(x)}$$

$n,k,N,K \in \{0,1,2,...\}$, and $\theta_L(N,K;n,k)$ is as in (1.7). Turning to the counter-example of Section 1, for an interval $I\subset\mathbb{R}$, introducing the notation

$$I_n = \int_I \phi^n(s)\, ds, \quad n = 0,1,2,...$$

and using commutation relations (2.3) we have for $n\geq 1$

$$B_{2n}^0(\chi_I)\, B_0^{2n}(\chi_I)\,\Phi = [B_{2n}^0(\chi_I), B_0^{2n}(\chi_I)]\,\Phi$$

$$= \sum_{L=1}^{2n}\binom{2n}{L}(2n)^{(L)}\, B_{2n-L}^{2n-L}(\phi^{L-1}\,\chi_I)\,\Phi$$

$$= \binom{2n}{2n}(2n)^{(2n)}\, B_0^0(\phi^{2n-1}\,\chi_I)\,\Phi = (2n)!\int_I \phi^{2n-1}(s)\, ds\,\Phi$$

and so

$$< B_{2n}^0(\chi_I)\, B_0^{2n}(\chi_I) >= (2n)! \int_I \phi^{2n-1}(s)\, ds = (2n)!\, I_{2n-1}$$

Similarly,

$$B_{2n}^0(\chi_I)\, (B_0^n(\chi_I))^2\, \Phi =$$

$$\left(B_0^n(\chi_I)\, B_{2n}^0(\chi_I) + [B_{2n}^0(\chi_I), B_0^n(\chi_I)]\right)\, B_0^n(\chi_I)\, \Phi$$

$$= B_0^n(\chi_I)\, B_{2n}^0(\chi_I)\, B_0^n(\chi_I)\, \Phi + [B_{2n}^0(\chi_I), B_0^n(\chi_I)]\, B_0^n(\chi_I)\, \Phi$$

$$= B_0^n(\chi_I)\, [B_{2n}^0(\chi_I), B_0^n(\chi_I)]\, \Phi + [B_{2n}^0(\chi_I), B_0^n(\chi_I)]\, B_0^n(\chi_I)\, \Phi$$

$$= B_0^n(\chi_I)\, \sum_{L=1}^{n} \hat{b}_L(2n,n)\, B_{2n-L}^{n-L}(\phi^{L-1}\, \chi_I)\, \Phi$$

$$+ \sum_{L=1}^{n} \hat{b}_L(2n,n)\, B_{2n-L}^{n-L}(\phi^{L-1}\, \chi_I)\, B_0^n(\chi_I)\, \Phi$$

$$= 0 + \sum_{L=1}^{n} \hat{b}_L(2n,n)\, [B_{2n-L}^{n-L}(\phi^{L-1}\, \chi_I), B_0^n(\chi_I)]\, \Phi$$

$$= \sum_{L_1=1}^{n} \sum_{L_2=1}^{n} \hat{b}_{L_1}(2n,n)\, \hat{b}_{L_2}(2n-L_1,n)\, B_{2n-(L_1+L_2)}^{2n-(L_1+L_2)}(\phi^{L_1+L_2-2}\, \chi_I)\, \Phi$$

$$= \hat{b}_n(2n,n)\, \hat{b}_n(n,n)\, B_0^0(\phi^{2n-2}\, \chi_I)\, \Phi = (2n)! \int_I \phi^{2n-2}(s)\, ds\, \Phi$$

which implies that

$$< B_{2n}^0(\chi_I)\, (B_0^n(\chi_I))^2 >= (2n)! \int_I \phi^{2n-2}(s)\, ds = (2n)!\, I_{2n-2}$$

We also have

$$B_n^0(\chi_I)\, (B_0^n(\chi_I))^2\, \Phi =$$

$$\left(B_0^n(\chi_I)\, B_n^0(\chi_I) + [B_n^0(\chi_I), B_0^n(\chi_I)]\right)\, B_0^n(\chi_I)\, \Phi$$

$$= B_0^n(\chi_I)\, B_n^0(\chi_I)\, B_0^n(\chi_I)\, \Phi + [B_n^0(\chi_I), B_0^n(\chi_I)]\, B_0^n(\chi_I)\, \Phi$$

$$= B_0^n(\chi_I)\, \left(B_0^n(\chi_I)\, B_n^0(\chi_I) + [B_n^0(\chi_I), B_0^n(\chi_I)]\right)\, \Phi + [B_n^0(\chi_I), B_0^n(\chi_I)]\, B_0^n(\chi_I)\, \Phi$$

$$= B_0^n(\chi_I)\,[B_n^0(\chi_I), B_0^n(\chi_I)]\,\Phi + [B_n^0(\chi_I), B_0^n(\chi_I)]\,B_0^n(\chi_I)\,\Phi$$

$$= B_0^n(\chi_I)\,\sum_{L=1}^{n}\hat{b}_L(n,n)\,B_{n-L}^{n-L}(\phi^{L-1}\,\chi_I)\,\Phi + \sum_{L=1}^{n}\hat{b}_L(n,n)\,B_{n-L}^{n-L}(\phi^{L-1}\,\chi_I)\,B_0^n(\chi_I)\,\Phi$$

$$= B_0^n(\chi_I)\,\hat{b}_n(n,n)\,B_0^0(\phi^{n-1}\,\chi_I)\,\Phi$$

$$+ \sum_{L=1}^{n}\hat{b}_L(n,n)\,\left(B_0^n(\chi_I)\,B_{n-L}^{n-L}(\phi^{L-1}\,\chi_I) + [B_{n-L}^{n-L}(\phi^{L-1}\,\chi_I), B_0^n(\chi_I)]\right)\Phi$$

$$= \hat{b}_n(n,n)\,\int_I \phi^{n-1}(s)\,ds\,B_0^n(\chi_I)\,\Phi + \hat{b}_n(n,n)\,B_0^n(\chi_I)\,B_0^0(\phi^{n-1}\,\chi_I)\,\Phi$$

$$+ \sum_{L_1=1}^{n}\sum_{L_2=1}^{n-L_1}\hat{b}_{L_1}(n,n)\,\hat{b}_{L_2}(n-L_1,n)\,B_{n-(L_1+L_2)}^{2n-(L_1+L_2)}(\phi^{L_1+L_2-2}\,\chi_I)\,\Phi$$

$$= 2\,\hat{b}_n(n,n)\,\int_I \phi^{n-1}(s)\,ds\,B_0^n(\chi_I)\,\Phi + \sum_{L=1}^{n-1}\hat{b}_L(n,n)\,\hat{b}_{n-L}(n-L,n)\,B_0^n(\phi^{n-2}\,\chi_I)\,\Phi$$

$$= 2\,(n!)\,\int_I \phi^{n-1}(s)\,ds\,B_0^n(\chi_I)\,\Phi + \left((2n)^{(n)} - 2\,(n!)\right)\,B_0^n(\phi^{n-2}\,\chi_I)\,\Phi$$

Thus

$$(B_n^0(\chi_I))^2\,(B_0^n(\chi_I))^2\,\Phi =$$

$$2\,(n!)\,I_{n-1}(s)\,B_n^0(\chi_I)\,B_0^n(\chi_I)\,\Phi + \left((2n)^{(n)} - 2\,(n!)\right)\,B_n^0(\chi_I)\,B_0^n(\phi^{n-2}\,\chi_I)\,\Phi$$

$$= 2\,(n!)\,I_{n-1}\,[B_n^0(\chi_I), B_0^n(\chi_I)]\,\Phi + \left((2n)^{(n)} - 2\,(n!)\right)\,[B_n^0(\chi_I), B_0^n(\phi^{n-2}\,\chi_I)]\,\Phi$$

$$= 2\,(n!)^2\,(I_{n-1})^2\,\Phi + \left((2n)^{(n)} - 2\,(n!)\right)\,\sum_{L=1}^{n}\hat{b}_L(n,n)\,B_{n-L}^{n-L}(\phi^{n-2+L-1}\,\chi_I)\,\Phi$$

$$= 2\,(n!)^2\,(I_{n-1})^2\,\Phi + \Big((2n)^{(n)} - 2\,(n!)\Big)\;\hat{b}_n(n,n)\,B_0^0(\phi^{2n-3}\,\chi_I)\,\Phi$$

$$= 2\,(n!)^2\,(I_{n-1})^2\,\Phi + \Big((2n)^{(n)} - 2\,(n!)\Big)\;(n!)\,I_{2n-3}\,\Phi$$

$$= 2\,(n!)^2\,(I_{n-1})^2\,\Phi + \big((2n)! - 2\,(n!)^2\big)\;I_{2n-3}\,\Phi$$

and so

$$< (B_n^0(\chi_I))^2\,(B_0^n(\chi_I))^2 >= 2\,(n!)^2\,(I_{n-1})^2 + \big((2n)! - 2\,(n!)^2\big)\;I_{2n-3}$$

Thus the matrix A of the counter-example of Section 1 has the form

$$A = \begin{bmatrix} < B_{2n}^0(\chi_I)\,B_0^{2n}(\chi_I) > & < B_{2n}^0(\chi_I)\,(B_0^n(\chi_I))^2 > \\ < B_{2n}^0(\chi_I)\,(B_0^n(\chi_I))^2 > & < (B_n^0(\chi_I))^2\,(B_0^n(\chi_I))^2 > \end{bmatrix}$$

$$= \begin{bmatrix} (2n)!\,I_{2n-1} & (2n)!\,I_{2n-2} \\ (2n)!\,I_{2n-2} & 2\,(n!)^2\,(I_{n-1})^2 + \big((2n)! - 2\,(n!)^2\big)\;I_{2n-3} \end{bmatrix}$$

with minor determinants

$$d_1 = (2n)!\,I_{2n-1}$$

which will be ≥ 0 if

$$I_{2n-1} \geq 0 \tag{2.4}$$

for all n and $I \subset \mathbb{R}$, and

$$d_2 = (2n)!\,(2\,(n!)^2\,I_{2n-1}\,(I_{n-1})^2$$

$$+\;\;((2n)! - 2\,(n!)^2)\,I_{2n-1}\,I_{2n-3} - (2n)!\,(I_{2n-2})^2)$$

which will be ≥ 0 if

$$2\,(n!)^2\,I_{2n-1}\,(I_{n-1})^2 + \big((2n)! - 2\,(n!)^2\big)\;I_{2n-1}\,I_{2n-3} - (2n)!\,(I_{2n-2})^2 \geq 0$$

i.e if

$$\big((2n)! - 2\,(n!)^2\big)\;I_{2n-1}\,I_{2n-3} \geq (2n)!\,(I_{2n-2})^2 - 2\,(n!)^2\,I_{2n-1}\,(I_{n-1})^2$$

which will be satisfied if

$$(I_{2n-2})^2 = I_{2n-1}\,I_{2n-3} \tag{2.5}$$

and

$$I_{2n-1}\,(I_{n-1})^2 \geq I_{2n-1}\,I_{2n-3} \tag{2.6}$$

for all n and $I \subset \mathbb{R}$. It was condition (2.6) that created all the trouble in the counter-example of Section 1.

3 A renormalization suggested by conditions (2.4)-(2.6)

We notice that if $\mathrm{supp}(\phi) \cap I = \emptyset$ then conditions (2.4)-(2.6) are trivially satisfied. If $\mathrm{supp}(\phi) \cap I \neq \emptyset$ then conditions (2.4)-(2.6) are satisfied by $I_n = 1$ for all $n = 1, 2, ...$, which is true if $\phi^n = \delta$ for all $n = 1, 2, ...$ The renormalization rule (2.1) then becomes

$$\delta^l(t-s) = \delta(s)\,\delta(t-s), \quad l = 2, 3, ... \tag{3.1}$$

and (1.2) takes the form

$$[b_t^{\dagger^n} b_t^k, b_s^{\dagger^N} b_s^K] = \tag{3.2}$$

$$\epsilon_{k,0}\epsilon_{N,0}\,(k\,N\,b_t^{\dagger^n}\,b_s^{\dagger^{N-1}}\,b_t^{k-1}\,b_s^k\,\delta(t-s)$$

$$+\sum_{L\geq 2}\binom{k}{L}N^{(L)}\,b_t^{\dagger^n}\,b_s^{\dagger^{N-L}}\,b_t^{k-L}\,b_s^K\,\delta(s)\,\delta(t-s))$$

$$-\epsilon_{K,0}\epsilon_{n,0}(K\,n\,b_s^{\dagger^N}\,b_t^{\dagger^{n-1}}\,b_s^{K-1}\,b_t^k\,\delta(t-s)$$

$$+\sum_{L\geq 2}\binom{K}{L}n^{(L)}\,b_s^{\dagger^N}\,b_t^{\dagger^{n-L}}\,b_s^{K-L}\,b_t^k\,\delta(s)\,\delta(t-s))$$

which, after multiplying both sides by $f(t)\bar{g}(s)$ and integrating the resulting identity, yields the commutation relations

$$[B_k^n(\bar{g}), B_K^N(f)] = (\epsilon_{k,0}\epsilon_{N,0}\,k\,N - \epsilon_{K,0}\epsilon_{n,0}\,K\,n)\;B_{K+k-1}^{N+n-1}(\bar{g}f) \tag{3.3}$$

$$+\sum_{L=2}^{(K\wedge n)\vee(k\wedge N)}\theta_L(n,k;N,K)\,\bar{g}(0)\,f(0)\,b_0^{\dagger^{N+n-l}}\,b_0^{K+k-l}$$

where $\theta_L(n,k;N,K)$ is as in (1.7). We can write (3.3) as

$$[B_k^n(\bar{g}), B_K^N(f)] = (\epsilon_{k,0}\epsilon_{N,0}\,k\,N - \epsilon_{K,0}\epsilon_{n,0}\,K\,n)\;B_{K+k-1}^{N+n-1}(\bar{g}f) \tag{3.4}$$

$$+\sum_{L=2}^{(K\wedge n)\vee(k\wedge N)} \theta_L(n,k;N,K)\, B_{K+k-L}^{N+n-L}(\bar{g}\, f\, \delta)$$

and notice that repeated commutations with the use of (3.4) will introduce terms containing $\delta(0)$.

4 The canonical RPQWN commutation relations

We may eliminate the singular terms from (3.3) by restricting to test functions f that satisfy $f(0)=0$. We then define the canonical RPQWN commutation relations as follows.

Definition 1. *For right-continuous step functions f,g such that $f(0)=g(0)=0$ we define*

$$[B_k^n(\bar{g}), B_K^N(f)]_R := (k\,N - K\,n)\ B_{k+K-1}^{n+N-1}(\bar{g}f) \tag{4.1}$$

Letting

$$C(n,k;N,K) := \begin{bmatrix} N & n \\ K & k \end{bmatrix} \tag{4.2}$$

commutation relations (4.1) can also be written as

$$[B_k^n(\bar{g}), B_K^N(f)]_R = \det C(n,k;N,K)\, B_{k+K-1}^{n+N-1}(\bar{g}f) \tag{4.3}$$

Proposition 1. *Commutation relations (4.1) define a Lie algebra.*

Proof. Clearly

$$[B_K^N(\bar{g}), B_K^N(f)]_R = 0$$

and

$$[B_K^N(\bar{g}), B_k^n(f)]_R = -[B_k^n(f), B_K^N(\bar{g})]_R$$

To show that commutation relations (4.1) satisfy the Jacobi identity we must show that (suppressing the test functions f and g) for all $n_i, k_i \geq 0$, where $i=1,2,3$,

$$\left[B_{k_1}^{n_1}, [B_{k_2}^{n_2}, B_{k_3}^{n_3}]_R\right]_R + \left[B_{k_3}^{n_3}, [B_{k_1}^{n_1}, B_{k_2}^{n_2}]_R\right]_R + \left[B_{k_2}^{n_2}, [B_{k_3}^{n_3}, B_{k_1}^{n_1}]_R\right]_R = 0$$

i.e. that

$$\det C(n_2,k_2;n_3,k_3)\ \det C(n_1,k_1;n_2+n_3-1,k_2+k_3-1)+$$

$$\det C(n_1,k_1;n_2,k_2)\ \det C(n_3,k_3;n_1+n_2-1,k_1+k_2-1)+$$

$$\det C(n_3,k_3;n_1,k_1)\ \det C(n_2,k_2;n_3+n_1-1,k_3+k_1-1)=0$$

which is the same as

$$(n_2k_3-n_3k_2)(n_1k_2+n_1k_3-n_1-k_1n_2-k_1n_3+k_1)+$$

$$(n_1k_2-n_2k_1)(n_3k_1+n_3k_2-n_3-n_1k_3-n_2k_3+k_3)+$$

$$(n_3k_1-n_1k_3)(n_2k_3+n_2k_1-n_2-n_3k_2-n_1k_2+k_2)=0$$

and is easily seen to be true.

$\square$

5 The w_∞ algebra

Definition 2. *The w_∞ algebra (see [4], [11]) is the infinite dimensional non-associative Lie algebra spanned by the generators $\hat{B}^n_k$, where $n,k\in\mathbb{Z}$ with $n\geq 2$, with commutation relations*

$$[\hat{B}^n_k,\hat{B}^N_K]_{w_\infty}=((N-1)\,k-(n-1)\,K)\ \hat{B}^{n+N-2}_{k+K} \tag{5.1}$$

and adjoint condition

$$\left(\hat{B}^n_k\right)^*=\hat{B}^n_{-k} \tag{5.2}$$

The w_∞ algebra is the basic algebraic structure of Conformal Field Theory in the study of quantum membranes. Since it contains as a sub algebra the Virasoro algebra with commutations

$$[\hat{B}^2_k(\bar{g}),\hat{B}^2_K(f)]_V:=(k-K)\,\hat{B}^2_{k+k}(\bar{g}f)$$

w_∞ can be viewed as an extended conformal algebra with an infinite number of additional symmetries (see [4]-[7], [11], [13], [14]). The elements of w_∞ are interpreted as area preserving diffeomorphisms of 2-manifolds. A quantum deformation of w_∞, called W_∞ and defined as a, large N, limit of Zamolodchikov's W_N algebra (see [14]), has been studied extensively (see [5]-[7], [11], [13]) in connection to two-dimensional Conformal Field Theory and Quantum Gravity. w_∞ is a "classical" or "Gel'fand-Dikii" algebra (see [9]) in the sense that it is a W algebra (see [11]) where all central terms are set to zero.

6 Poisson brackets

The construction produced in the following section was inspired by the analogy with the realization of the w–algebra in terms of Poisson brackets. This realization is well known and, in the following, we recall it briefly.

Definition 3. *For scalar-valued differentiable functions $f(x, y)$ and $g(x, y)$, the Poisson bracket $\{f, g\}$ is defined by*

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

We notice that the functions $f(x, y) = x$ and $g(x, y) = y$ satisfy $\{f, g\} = 1$ which we can write as

$$\{x, y\} = 1$$

in analogy with the Canonical Commutation Relations (CCR). We can model commutation relations (5.1) and the adjoint condition (5.2) using the Poisson bracket as follows:

Proposition 2. *For $n, k \in \mathbb{Z}$ with $n \geq 2$, let $f_{n,k} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be defined by $f_{n,k}(x, y) = e^{ikx}\, y^{n-1}$. Then*

$$\{f_{n,k}(x, y), f_{N,K}(x, y)\} = i\ (k(N-1) - K(n-1))\ f_{n+N-2,k+K}(x, y) \tag{6.1}$$

and

$$\overline{f}_{n,k}(x, y) = f_{n,-k}(x, y)$$

Proof. By the definition of the Poisson bracket,

$$\begin{aligned}\{f_{n,k}(x, y), f_{N,K}(x, y)\} &= \frac{\partial}{\partial x}(e^{ikx}\, y^{n-1})\, \frac{\partial}{\partial y}(e^{iKx}\, y^{N-1}) \\ &\quad - \frac{\partial}{\partial y}(e^{ikx}\, y^{n-1})\, \frac{\partial}{\partial x}(e^{iKx}\, y^{N-1}) \\ &= i\ (k(N-1) - K(n-1))\ e^{i(k+K)x}\, y^{n+N-3} \\ &= i\ (k(N-1) - K(n-1))\ f_{n+N-2,k+K}(x, y)\end{aligned}$$

Moreover,

$$\overline{f}_{n,k}(x, y) = \overline{e^{ikx}\, y^{n-1}} = e^{-ikx}\, y^{n-1} = f_{n,-k}(x, y)$$

□

Using the prescription

$$[A, B] = \frac{\hbar}{i} \{A, B\}$$

we thus obtain, letting $\hbar = 1$, that the quantized version of (6.1) is

$$[f_{n,k}, f_{N,K}] = (k(N-1) - K(n-1))\ f_{n+N-2,k+K}$$

which is precisely (5.1). Similarly, we can model commutation relations (4.1) and the RPQWN adjoint condition $(B^n_k)^* = B^k_n$ using the Poisson bracket as follows:

Proposition 3. *For $n, k \geq 0$, let $g_{n,k} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be defined by*

$$g_{n,k}(x, y) = \left(\frac{x+iy}{\sqrt{2}}\right)^n \left(\frac{x-iy}{\sqrt{2}}\right)^k$$

Then

$$\{g_{n,k}(x, y), g_{N,K}(x, y)\} = i\ (kN - nK)\ g_{n+N-1,k+K-1}(x, y) \tag{6.2}$$

and

$$\overline{g}_{n,k}(x, y) = g_{k,n}(x, y)$$

Proof. By the definition of the Poisson bracket,

$$\{g_{n,k}(x, y), g_{N,K}(x, y)\} =$$

$$\frac{\partial}{\partial x}\left(\left(\frac{x+iy}{\sqrt{2}}\right)^n \left(\frac{x-iy}{\sqrt{2}}\right)^k\right) \frac{\partial}{\partial y}\left(\left(\frac{x+iy}{\sqrt{2}}\right)^N \left(\frac{x-iy}{\sqrt{2}}\right)^K\right)$$

$$-\frac{\partial}{\partial y}\left(\left(\frac{x+iy}{\sqrt{2}}\right)^n \left(\frac{x-iy}{\sqrt{2}}\right)^k\right) \frac{\partial}{\partial x}\left(\left(\frac{x+iy}{\sqrt{2}}\right)^N \left(\frac{x-iy}{\sqrt{2}}\right)^K\right)$$

$$= i\ (kN - nK)\ 2^{1-\frac{n+k+N+K}{2}}\ (x+iy)^{n+N-1}(x-iy)^{k+K-1}$$

$$= i\ (kN - nK) \left(\frac{x+iy}{\sqrt{2}}\right)^{n+N-1} \left(\frac{x-iy}{\sqrt{2}}\right)^{k+K-1}$$

$$= i\ (kN - nK)\ g_{n+N-1,k+K-1}(x, y)$$

Moreover,

$$\overline{g}_{n,k}(x, y) = \overline{\left(\frac{x+iy}{\sqrt{2}}\right)^n \left(\frac{x-iy}{\sqrt{2}}\right)^k} = \left(\frac{x-iy}{\sqrt{2}}\right)^n \left(\frac{x+iy}{\sqrt{2}}\right)^k = g_{k,n}(x, y)$$

□

We therefore have, as above, that the quantized version of (6.2) is

$$[g_{n,k}, g_{N,K}] = (kN - nK)\; g_{n+N-1,k+K-1}$$

which is (4.1).

7 White noise form of the w_∞ generators and commutation relations

Motivated by the results of the previous section we introduce the following:

Definition 4. *For right-continuous step functions f, g such that $f(0) = g(0) = 0$, and for $n, k \in \mathbb{Z}$ with $n \geq 2$, we define*

$$\hat{B}^n_k(f) := \int_{\mathbb{R}} f(t)\, e^{\frac{k}{2}(b_t - b_t^\dagger)} \left(\frac{b_t + b_t^\dagger}{2}\right)^{n-1} e^{\frac{k}{2}(b_t - b_t^\dagger)}\, dt \tag{7.1}$$

with involution

$$\left(\hat{B}^n_k(f)\right)^* = \hat{B}^n_{-k}(\bar{f})$$

In particular,

$$\hat{B}^2_k(f) := \int_{\mathbb{R}} f(t)\, e^{\frac{k}{2}(b_t - b_t^\dagger)} \left(\frac{b_t + b_t^\dagger}{2}\right) e^{\frac{k}{2}(b_t - b_t^\dagger)}\, dt \tag{7.2}$$

is the RPQWN form of the Virasoro operators

The integral on the right hand side of (7.1) is meant in the sense that one expands the exponential series (resp. the power), applies the commutation relations (1.2) to bring the resulting expression to normal order, introduces the renormalization prescription (3.1), integrates the resulting expressions after multiplication by a test function and interprets the result as a quadratic form on the exponential vectors.

Lemma 2. *Let x , D and h be three operators satisfying the Heisenberg commutation relations*

$$[D, x] = h, \;\; [D, h] = [x, h] = 0$$

Then, for all $s, a, c \in \mathbb{C}$

$$e^{s(x+aD+ch)} = e^{sx} e^{saD} e^{(sc + \frac{s^2 a}{2})h}$$

$$e^{sD} e^{ax} = e^{ax} e^{sD} e^{ash}$$

and for all $n, m \in \mathbb{N}$

$$D^n x^m = \sum_{j=1}^{n\wedge m} \binom{n,m}{j} x^{m-j} D^{n-j} h^j$$

where

$$\binom{n,m}{j} = \binom{n}{j}\binom{m}{j} j!$$

Proof. This is just a combination of Propositions 2.2.2, 2.2.1 and 4.1.1 of [8]. $\square$

Lemma 3. *In the notation of lemma 2, for all $\lambda \in \{0,1,...\}$ and $a \in \mathbb{C}$*

$$D^\lambda e^{ax} = e^{ax} \sum_{m=0}^{\lambda} \binom{\lambda}{m} D^m (ah)^{\lambda-m}$$

and

$$e^{sD} x^\lambda = \sum_{m=0}^{\lambda} \binom{\lambda}{m} x^m (sh)^{\lambda-m} e^{sD}$$

Proof. By lemma 2

$$D^\lambda e^{ax} = \frac{\partial^\lambda}{\partial s^\lambda}|_{s=0}\left(e^{sD} e^{ax}\right) = \frac{\partial^\lambda}{\partial s^\lambda}|_{s=0}\left(e^{ax} e^{sD} e^{ash}\right) = e^{ax} \frac{\partial^\lambda}{\partial s^\lambda}|_{s=0}\left(e^{sD} e^{ash}\right)$$

$$= e^{ax} \sum_{m=0}^{\lambda} \binom{\lambda}{m} \frac{\partial^m}{\partial s^m}|_{s=0}\left(e^{sD}\right) \frac{\partial^{\lambda-m}}{\partial s^{\lambda-m}}|_{s=0}\left(e^{ash}\right) = e^{ax} \sum_{m=0}^{\lambda} \binom{\lambda}{m} D^m (ah)^{\lambda-m}$$

Similarly,

$$e^{sD} x^\lambda = \frac{\partial^\lambda}{\partial a^\lambda}|_{a=0}\left(e^{sD} e^{ax}\right) = \frac{\partial^\lambda}{\partial a^\lambda}|_{a=0}\left(e^{ax} e^{sD} e^{ash}\right) = \frac{\partial^\lambda}{\partial a^\lambda}|_{a=0}\left(e^{ax} e^{ash}\right) e^{sD}$$

$$= \sum_{m=0}^{\lambda} \binom{\lambda}{m} \frac{\partial^m}{\partial a^m}|_{a=0}\left(e^{ax}\right) \frac{\partial^{\lambda-m}}{\partial a^{\lambda-m}}|_{a=0}\left(e^{ash}\right) e^{sD} = \sum_{m=0}^{\lambda} \binom{\lambda}{m} x^m (sh)^{\lambda-m} e^{sD}$$

$\square$

Lemma 4. *Let the exponential and powers of white noise be interpreted as described in Definition (4). Then:*

(i) For fixed $t, s \in \mathbb{R}$, the operators $D = b_t - b_t^\dagger$, $x = b_s + b_s^\dagger$ and $h = 2\,\delta(t-s)$ satisfy the commutation relations of lemma 2.

(ii) For fixed $t, s \in \mathbb{R}$, the operators $D = b_t + b_t^\dagger$, $x = b_s - b_s^\dagger$ and $h = -2\,\delta(t-s)$ satisfy the commutation relations of lemma 2.

Proof. To prove (i) we notice that

$$[D,x]=[b_t-b_t^\dagger,b_s+b_s^\dagger]=[b_t,b_s^\dagger]-[b_t^\dagger,b_s]=[b_t,b_s^\dagger]+[b_s,b_t^\dagger]=\delta(t-s)+\delta(s-t)=h$$

while, clearly, $[D,h]=[x,h]=0$. The proof of (ii) is similar.

□

Proposition 4. *If f,g are right-continuous step functions such that $f(0)=g(0)=0$ and the powers of the delta function are renormalized by the prescription (3.1), then*

$$[\hat{B}^n_k(\bar{g}),\hat{B}^N_K(f)]=(k\,(N-1)-K\,(n-1))\;\hat{B}^{n+N-2}_{k+K}(\bar{g}f) \tag{7.3}$$

i.e the operators $\hat{B}^n_k$ of Definition 4 satisfy the commutation relations of the w_∞ algebra. In particular,

$$[\hat{B}^2_k(\bar{g}),\hat{B}^2_K(f)]=(k-K)\;\hat{B}^2_{k+K}(\bar{g}f) \tag{7.4}$$

i.e the operators $\hat{B}^2_k$ of Definition 4 satisfy the commutation relations of the Virasoro algebra. Here $[x,y]:=xy-yx$ is the usual operator commutator.

Proof. To prove (7.3), we notice that by Definition 4, its left hand side is

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{g}(t)f(s)[e^{\frac{k}{2}(b_t-b_t^\dagger)}\left(\frac{b_t+b_t^\dagger}{2}\right)^{n-1}e^{\frac{k}{2}(b_t-b_t^\dagger)},$$

$$e^{\frac{K}{2}(b_s-b_s^\dagger)}\left(\frac{b_s+b_s^\dagger}{2}\right)^{N-1}e^{\frac{K}{2}(b_s-b_s^\dagger)}]\,dt\,ds$$

$$=\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{g}(t)f(s)\,e^{\frac{k}{2}(b_t-b_t^\dagger)}\left(\frac{b_t+b_t^\dagger}{2}\right)^{n-1}e^{\frac{k}{2}(b_t-b_t^\dagger)}$$

$$\times e^{\frac{K}{2}(b_s-b_s^\dagger)}\left(\frac{b_s+b_s^\dagger}{2}\right)^{N-1}e^{\frac{K}{2}(b_s-b_s^\dagger)}\,dt\,ds$$

$$-\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{g}(t)f(s)\,e^{\frac{K}{2}(b_s-b_s^\dagger)}\left(\frac{b_s+b_s^\dagger}{2}\right)^{N-1}e^{\frac{K}{2}(b_s-b_s^\dagger)}$$

$$\times e^{\frac{k}{2}(b_t-b_t^\dagger)}\left(\frac{b_t+b_t^\dagger}{2}\right)^{n-1}e^{\frac{k}{2}(b_t-b_t^\dagger)}\,dt\,ds$$

which, since $[b_t-b_t^\dagger,b_s+b_s^\dagger]=0$, is

$$=\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{g}(t)f(s)\,e^{\frac{k}{2}(b_t-b_t^\dagger)}\left(\frac{b_t+b_t^\dagger}{2}\right)^{n-1}e^{\frac{K}{2}(b_s-b_s^\dagger)}$$

$$\times e^{\frac{k}{2}(b_t-b_t^\dagger)} \left(\frac{b_s+b_s^\dagger}{2}\right)^{N-1} e^{\frac{K}{2}(b_s-b_s^\dagger)}\,dt\,ds$$

$$-\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{g}(t)f(s)\,e^{\frac{K}{2}(b_s-b_s^\dagger)} \left(\frac{b_s+b_s^\dagger}{2}\right)^{N-1} e^{\frac{k}{2}(b_t-b_t^\dagger)}$$

$$\times e^{\frac{K}{2}(b_s-b_s^\dagger)} \left(\frac{b_t+b_t^\dagger}{2}\right)^{n-1} e^{\frac{k}{2}(b_t-b_t^\dagger)}\,dt\,ds$$

$$= \frac{1}{2^{n+N-2}}\{\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{g}(t)f(s)\,e^{\frac{k}{2}(b_t-b_t^\dagger)}(b_t+b_t^\dagger)^{n-1}e^{\frac{K}{2}(b_s-b_s^\dagger)}$$

$$\times e^{\frac{k}{2}(b_t-b_t^\dagger)}(b_s+b_s^\dagger)^{N-1}e^{\frac{K}{2}(b_s-b_s^\dagger)}\,dt\,ds$$

$$-\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{g}(t)f(s)\,e^{\frac{K}{2}(b_s-b_s^\dagger)}(b_s+b_s^\dagger)^{N-1}e^{\frac{k}{2}(b_t-b_t^\dagger)}e^{\frac{K}{2}(b_s-b_s^\dagger)}$$

$$\times(b_t+b_t^\dagger)^{n-1}e^{\frac{k}{2}(b_t-b_t^\dagger)}\,dt\,ds\}$$

Since, by lemmas 3 and 4,

$$e^{\frac{K}{2}(b_s-b_s^\dagger)}(b_t+b_t^\dagger)^{n-1} =$$

$$\sum_{m=0}^{n-1}\binom{n-1}{m}(b_t+b_t^\dagger)^m K^{n-1-m}\delta^{n-1-m}(t-s)\,e^{\frac{K}{2}(b_s-b_s^\dagger)}$$

and

$$e^{\frac{k}{2}(b_t-b_t^\dagger)}(b_s+b_s^\dagger)^{N-1} =$$

$$\sum_{m=0}^{N-1}\binom{N-1}{m}(b_s+b_s^\dagger)^m k^{N-1-m}\delta^{N-1-m}(t-s)\,e^{\frac{k}{2}(b_t-b_t^\dagger)}$$

and

$$(b_t+b_t^\dagger)^{n-1}e^{\frac{K}{2}(b_s-b_s^\dagger)} =$$

$$e^{\frac{K}{2}(b_s-b_s^\dagger)}\sum_{m=0}^{n-1}\binom{n-1}{m}(b_t+b_t^\dagger)^m K^{n-1-m}(-1)^{n-1-m}\delta^{n-1-m}(t-s)$$

and

$$(b_s+b_s^\dagger)^{N-1}e^{\frac{k}{2}(b_t-b_t^\dagger)} =$$

$$e^{\frac{k}{2}(b_t-b_t^\dagger)}\sum_{m=0}^{N-1}\binom{N-1}{m}(b_s+b_s^\dagger)^m k^{N-1-m}(-1)^{N-1-m}\delta^{N-1-m}(t-s)$$

we find that

$$[\hat{B}_k^n(\bar{g}),\hat{B}_K^N(f)]=\frac{1}{2^{n+N-2}}\{\sum_{m_1=0}^{n-1}\sum_{m_2=0}^{N-1}\binom{n-1}{m_1}\binom{N-1}{m_2}$$

$$\times(-1)^{n-1-m_1}K^{n-1-m_1}k^{N-1-m_2}$$

$$\times\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{g}(t)f(s)\,e^{\frac{k}{2}(b_t-b_t^\dagger)}e^{\frac{K}{2}(b_s-b_s^\dagger)}$$

$$\times(b_t+b_t^\dagger)^{m_1}(b_s+b_s^\dagger)^{m_2}e^{\frac{k}{2}(b_t-b_t^\dagger)}e^{\frac{K}{2}(b_s-b_s^\dagger)}$$

$$\times\delta^{n-1-m_1}(t-s)\delta^{N-1-m_2}(t-s)\,dt\,ds$$

$$-\sum_{m_3=0}^{N-1}\sum_{m_4=0}^{n-1}\binom{N-1}{m_3}\binom{n-1}{m_4}(-1)^{N-1-m_3}k^{N-1-m_3}K^{n-1-m_4}$$

$$\times\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{g}(t)f(s)\,e^{\frac{K}{2}(b_s-b_s^\dagger)}e^{\frac{k}{2}(b_t-b_t^\dagger)}$$

$$\times(b_s+b_s^\dagger)^{m_3}(b_t+b_t^\dagger)^{m_4}e^{\frac{K}{2}(b_s-b_s^\dagger)}e^{\frac{k}{2}(b_t-b_t^\dagger)}$$

$$\times\delta^{N-1-m_3}(t-s)\delta^{n-1-m_4}(t-s)\,dt\,ds\}$$

The case ($m_1=n-1$, $m_2=N-1$) cancels out with ($m_3=N-1$, $m_4=n-1$). By the renormalization prescription (3.1) and the choice of test functions that vanish at zero, the terms $\sum_{m_1=0}^{n-3}\sum_{m_2=0}^{N-3}$ and $\sum_{m_3=0}^{N-3}\sum_{m_4=0}^{n-3}$ are equal to zero. The only surviving terms are ($m_1=n-1$, $m_2=N-2$), ($m_1=n-2$, $m_2=N-1$) , ($m_3=N-1$, $m_4=n-2$) and ($m_3=N-2$, $m_4=n-1$) and we obtain

$$[\hat{B}_k^n(\bar{g}),\hat{B}_K^N(f)]=$$

$$=\frac{1}{2^{n+N-2}}((N-1)k-(n-1)K-(n-1)K+(N-1)k)$$

$$\times\int_{\mathbb{R}}\bar{g}(t)f(t)\,e^{\frac{k+K}{2}(b_t-b_t^\dagger)}(b_t+b_t^\dagger)^{n+N-3}e^{\frac{k+K}{2}(b_t-b_t^\dagger)}\,dt$$

$$=\frac{2}{2^{n+N-2}}((N-1)k-(n-1)K)$$

$$\times \int_{\mathbb{R}} \bar{g}(t)f(t)\, e^{\frac{k+K}{2}(b_t-b_t^\dagger)}(b_t+b_t^\dagger)^{n+N-3}e^{\frac{k+K}{2}(b_t-b_t^\dagger)}\, dt$$

$$= \frac{1}{2^{n+N-3}}((N-1)k-(n-1)K)$$

$$\times \int_{\mathbb{R}} \bar{g}(t)f(t)\, e^{\frac{k+K}{2}(b_t-b_t^\dagger)}(b_t+b_t^\dagger)^{n+N-3}e^{\frac{k+K}{2}(b_t-b_t^\dagger)}\, dt$$

$$= (k(N-1)-K(n-1))\hat{B}_{k+K}^{n+N-2}(\bar{g}f)$$

The proof of (7.4) follows from (7.3) by letting $n=N=2$. □

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